

XXV. *A new Method of finding Fluents by Continuation.* By
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ART. I. Put $\dot{F} = \frac{x^m \dot{x}}{a^n + x^n} = x^{m-n} \dot{x} - a^n x^{m-2n} \dot{x} + a^{2n} x^{m-3n} \dot{x} -$
 $\&c. \pm \frac{a^{2n} x^{m-2n} \dot{x}}{a^n + x^n}$, then $F = \frac{x^{m-n+1}}{m-n+1} - \frac{a^n x^{m-2n+1}}{m-2n+1} + \frac{a^{2n} x^{m-3n+1}}{m-3n+1} - \&c.$
 $\pm W$, where W represents the fluent of the last term. Now
 $\frac{x^m \dot{x}}{(a^n + x^n)^r} = \frac{x^m \dot{x} \times (a^n + x^n)^{1-r}}{a^n + x^n}$; hence $\int \frac{x^m \dot{x}}{(a^n + x^n)^r} = \int \frac{x^m \dot{x} \times (a^n + x^n)^{1-r}}{a^n + x^n} = F$
 $\times (a^n + x^n)^{1-r} - \int F \times \frac{1-r \cdot n x^{n-1} \dot{x}}{a^n + x^n} =$ (by substituting for F its
value in the latter quantity) $F \times (a^n + x^n)^{1-r} - \frac{1-r \cdot n}{m-n+1} \times$
 $\int \frac{x^m \dot{x}}{a^n + x^n} + \frac{1-r \cdot n a^n}{m-2n+1} \times \int \frac{x^{m-n} \dot{x}}{a^n + x^n} - \frac{1-r \cdot n a^{2n}}{m-3n+1} \times \int \frac{x^{m-2n} \dot{x}}{a^n + x^n} + \&c. \pm$
 $\int W \times \frac{1-r \cdot n x^{n-1} \dot{x}}{a^n + x^n}$; transpose $-\frac{1-r \cdot n}{m-n+1} \times \int \frac{x^m \dot{x}}{a^n + x^n}$ and divide
both sides of the equation by $\frac{m-rn+1}{m-n+1}$ and we have
 $\int \frac{x^m \dot{x}}{a^n + x^n} = \frac{m-n+1}{m-rn+1} \times F \times (a^n + x^n)^{1-r} + \frac{m-n+1}{m-rn+1} \times \frac{1-r \cdot n a^n}{m-2n+1} \times$
 $\int \frac{x^{m-n} \dot{x}}{a^n + x^n} - \frac{m-n+1}{m-rn+1} \times \frac{1-r \cdot n a^{2n}}{m-3n+1} \times \int \frac{x^{m-2n} \dot{x}}{a^n + x^n} + \&c. \pm \frac{m-n+1}{m-rn+1} \times$
 \int

$\int W \times \frac{\overline{1-r} \cdot n x^{n-1} \dot{x}}{a^n + x^n} r$; now the fluent of the last term is

$$= \frac{m-n+1}{m-rn+1} \times W \times \overline{a^n + x^n}^{1-r} = \frac{m-n+1}{m-rn+1} \times \int \frac{a^{2n} x^{m-2n} \dot{x}}{a^n + x^n} r$$
; hence by

substituting this quantity for the last term, it is manifest, that the first part $= \frac{m-n+1}{m-rn+1} \times W \times \overline{a^n + x^n}^{1-r}$ will be destroyed

by the last term of $\frac{m-n+1}{m-rn+1} \times F \times \overline{a^n + x^n}^{1-r}$, when we substitute

for F its value; hence if we put $M = \frac{x^{m-n+1}}{m-n+1} - \frac{a^n x^{m-2n-1}}{m-2n+1} + \frac{a^{2n} x^{m-3n+1}}{m-3n+1} - \&c.$ omitting the last term $\pm W$, we have

$$\int \frac{x^{m-n} \dot{x}}{a^n + x^n} r = \frac{m-n+1}{m-rn+1} \times M \times \overline{a^n + x^n}^{1-r} + \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r} \cdot n a^n}{m-2n+1} \times$$

$$\int \frac{x^{m-n} \dot{x}}{a^n + x^n} r = \frac{m-n+1}{m-rn+1} \times \frac{\overline{1-r} \cdot n a^{2n}}{m-3n+1} \times \int \frac{x^{m-2n} \dot{x}}{a^n + x^n} r + \&c. = \frac{m-n+1}{m-rn+1} \times$$

$$\times a^{2n} \times \int \frac{x^{m-2n} \dot{x}}{a^n + x^n} r$$
; hence, if the fluent of the last term be

given, we have the general law of continuation by which we

may find the fluent of $\frac{x^m \dot{x}}{a^n + x^n} r$. If the fluxion be $\frac{x^m \dot{x}}{a^n + x^n} r$ all the

terms after the first will be negative, and the last always positive.

Ex. 1. Given the fluent of $\frac{\dot{x}}{\sqrt{1+x^2}}$ to find the fluent of

$$\frac{x^{21} \dot{x}}{\sqrt{1+x^2}}$$

Here $n=2$, $a=1$, $m=25$, $r=\frac{1}{2}$, $M = \frac{x^{25-1}}{25-1} - \frac{x^{25-3}}{25-3} + \&c.$ to

$\pm x$,

$\pm x$, and the fluent of $\frac{\dot{x}}{\sqrt{1+x^2}}$ is the hyp. log. $x + \sqrt{1+x^2}$

which call Q; hence $\int \frac{x^{2s}\dot{x}}{\sqrt{1+x^2}} = \frac{2s-1}{2s} \times M \times \sqrt{1+x^2} + \frac{2s-1}{2s \cdot 2s-3}$

$$\times \int \frac{x^{2s-2}\dot{x}}{\sqrt{1+x^2}} - \frac{2s-1}{2s \cdot 2s-5} \times \int \frac{x^{2s-4}\dot{x}}{\sqrt{1+x^2}} + \&c. \pm \frac{2s-1}{2s} \times Q.$$

$$\text{If } s=1, \int \frac{x^2\dot{x}}{\sqrt{1+x^2}} = \frac{1}{2}\sqrt{1+x^2} \times x - \frac{1}{2}Q = \alpha.$$

$$s=2, \int \frac{x^4\dot{x}}{\sqrt{1+x^2}} = \frac{3}{4}\sqrt{1+x^2} \times \frac{x^3}{3} - x + \frac{1}{4}\alpha + \frac{3}{4}Q = \beta.$$

$$s=3, \int \frac{x^6\dot{x}}{\sqrt{1+x^2}} = \frac{5}{6}\sqrt{1+x^2} \times \frac{x^5}{5} - \frac{x^3}{3} + x + \frac{5}{6}\beta - \frac{5}{6}\alpha - \frac{5}{6}Q = \gamma.$$

$$s=4, \int \frac{x^8\dot{x}}{\sqrt{1+x^2}} = \frac{7}{8}\sqrt{1+x^2} \times \frac{x^7}{7} - \frac{x^5}{5} + \frac{x^3}{3} - x + \frac{7}{8}\gamma - \frac{7}{8}\beta$$

$$+ \frac{7}{8}\alpha + \frac{7}{8}Q.$$

&c.

&c.

&c.

Ex. 2. To find the fluent of $x^{\frac{s}{2}}\dot{x}\sqrt{2+x}$, given the fluent of $x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}$, and s an odd number.

Here $a=2$, $n=1$, $r=-\frac{1}{2}$, $\frac{s}{2}=m$, $m-vn=-\frac{1}{2}$, or $\frac{s}{2}-v$

$$=-\frac{1}{2}; \therefore v=\frac{s+1}{2}, M=\frac{2x^{\frac{s}{2}}}{s} - \frac{4x^{\frac{s}{2}-1}}{s-2} + \frac{8x^{\frac{s}{2}-2}}{s-4} - \&c. \text{ and the}$$

fluent (Q) of $x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}$ is $\pi + \sqrt{2x+x^2}$, where π =hyp.

log. $1+x+\sqrt{2x+x^2}$; hence $\int x^{\frac{s}{2}}\dot{x}\sqrt{2+x} = \frac{s}{s+3} \times \sqrt{2+x}^{\frac{3}{2}} \times$

$$M + \frac{s}{s+3} \times \frac{b}{s-2} \times \int x^{\frac{s}{2}-1}\dot{x}\sqrt{2+x} - \frac{s}{s+3} \times \frac{12}{s-4} \times \int x^{\frac{s}{2}-2}\dot{x}\sqrt{2+x}$$

$$+ \frac{s}{s+3} \times \frac{24}{s-6} \times \int x^{\frac{s}{2}-3}\dot{x}\sqrt{2+x} \&c. \pm \frac{s}{s+3} \times 2^{\frac{s+1}{2}} \times \int x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}.$$

If

$$\text{If } s = 1, \int x^{\frac{1}{2}} \dot{x} \sqrt{2+x} = \frac{1}{4} \times \overline{2+x}^{\frac{3}{2}} \times 2x^{\frac{1}{2}} - \frac{1}{2} Q = \alpha.$$

$$s = 3, \int x^{\frac{3}{2}} \dot{x} \sqrt{2+x} = \frac{1}{2} \times \overline{2+x}^{\frac{3}{2}} \times \frac{2x^{\frac{3}{2}}}{3} - 4x^{\frac{1}{2}} + 3\alpha + 2Q = \beta.$$

$$s = 5, \int x^{\frac{5}{2}} \dot{x} \sqrt{2+x} = \frac{5}{8} \times \overline{2+x}^{\frac{3}{2}} \times \frac{2x^{\frac{5}{2}}}{5} - \frac{4x^{\frac{3}{2}}}{3} + 8x^{\frac{1}{2}} + \frac{5}{4} \beta -$$

$$\frac{15}{2} \alpha - 5Q = \gamma.$$

$$s = 7, \int x^{\frac{7}{2}} \dot{x} \sqrt{2+x} = \frac{7}{10} \times \overline{2+x}^{\frac{3}{2}} \times \frac{2x^{\frac{7}{2}}}{7} - \frac{4x^{\frac{5}{2}}}{5} + \frac{8x^{\frac{3}{2}}}{3} - 16x^{\frac{1}{2}} +$$

$$\frac{21}{25} \gamma - \frac{14}{5} \beta + \frac{84}{5} \alpha + \frac{56}{5} Q.$$

&c.

&c.

&c.

$$\text{II. Let } \dot{F} \frac{x^n \dot{x}}{a + bx^m + x^{2m}} = x^{n-2m} \dot{x} - Px^{n-3m} \dot{x} + Qx^{n-4m} \dot{x} - \&c. \pm$$

$$\frac{Vx^{n-sm} \dot{x}}{a + bx^m + x^{2m}} \pm \frac{Wx^{n-1+1 \cdot m} \dot{x}}{a + bx^m + x^{2m}}, \text{ then } F = \frac{x^{n-2m+1}}{n-2m+1} - \frac{Px^{n-3m+1}}{n-3m+1} + \frac{Qx^{n-4m+1}}{n-4m+1}$$

- &c. $\pm T \pm U$, where T and U are put for the fluents of the two last terms, and P, Q, &c. for the co-efficients arising from

the division. Now, $\int \frac{x^n \dot{x}}{\sqrt{a + bx^m + x^{2m}}} = \int \frac{x^n \dot{x} \times \overline{a + bx^m + x^{2m}}^{1-r}}{a + bx^m + x^{2m}} =$

$$F \times \overline{a + bx^m + x^{2m}}^{1-r} - \int F + \frac{1-r \cdot mbx^{m-1} \dot{x} + 1-r \cdot 2mx^{2m-1} \dot{x}}{\overline{a + bx^m + x^{2m}}^r} = (\text{by}$$

substituting for F its value in the latter quantity, and putting A, B, C, &c for the co-efficients which arise in consequence

thereof) $F \times \overline{a + bx^m + x^{2m}}^{1-r} - A \times \int \frac{x^n \dot{x}}{\overline{a + bx^m + x^{2m}}^r} + B \times$

$$\int \frac{x^{n-m} \dot{x}}{\overline{a + bx^m + x^{2m}}^r} - C \times \int \frac{x^{n-2m} \dot{x}}{\overline{a + bx^m + x^{2m}}^r} + \&c.$$

$$= \int T \times \frac{\overline{1-r} \cdot m b x^{m-1} \dot{x} + \overline{1-r} \cdot 2 m x^{2m-1} \dot{x}}{a + b x^m + x^{2m}}^r$$

$$= \int U \times \frac{\overline{1-r} \cdot m b x^{m-1} \dot{x} + \overline{1-r} \cdot 2 m x^{2m-1} \dot{x}}{a + b x^m + x^{2m}}^r; \text{ hence by transposition}$$

and division we have $\int \frac{x^n \dot{x}}{a + b x^m + x^{2m}}^r = \frac{1}{1+A} \times F \times \overline{a + b x^m + x^{2m}}^{1-r}$

$$+ \frac{B}{1+A} \times \int \frac{x^{n-m} \dot{x}}{a + b x^m + x^{2m}}^r - \frac{C}{1+A} \times \int \frac{x^{n-2m} \dot{x}}{a + b x^m + x^{2m}}^r + \&c.$$

$$= \int \frac{T}{1+A} \times \frac{\overline{1-r} \cdot m b x^{m-1} \dot{x} + \overline{1-r} \cdot 2 m x^{2m-1} \dot{x}}{a + b x^m + x^{2m}}^r$$

$$= \int \frac{U}{1+A} \times \frac{\overline{1-r} \cdot m b x^{m-1} \dot{x} + \overline{1-r} \cdot 2 m x^{2m-1} \dot{x}}{a + b x^m + x^{2m}}^r. \text{ Now the fluents}$$

of these two last terms are $= \frac{T}{1+A} \times \overline{a + b x^m + x^{2m}}^{1-r} = \frac{V}{1+A}$

$$\int \frac{x^{n-3m} \dot{x}}{a + b x^m + x^{2m}}^r \text{ and } = \frac{U}{1+A} \times \overline{a + b x^m + x^{2m}}^{1-r} = \frac{W}{1+A} \int \frac{x^{n-3+1 \cdot m} \dot{x}}{a + b x^m + x^{2m}}^r$$

respectively; hence, by substituting these for the last term,

it is manifest that $= \frac{T}{1+A} \times \overline{a + b x^m + x^{2m}}^{1-r}$ and $= \frac{U}{1+A}$

$\times \overline{a + b x^m + x^{2m}}^{1-r}$ will be destroyed by the two last terms of

$\frac{1}{1+A} \times F \times \overline{a + b x^m + x^{2m}}^{1-r}$ when we substitute for F its value;

hence, if we put $M = \frac{x^{n-2m-1}}{n-2m+1} - \frac{P x^{n-3m+1}}{n-3m+1} + \frac{Q x^{n-4m+1}}{n-4m+1} - \&c.$

omitting the two last terms $\pm T$ and $\pm U$, we shall have

$$\int \frac{x^n \dot{x}}{a + b x^m + x^{2m}}^r = \frac{1}{1+A} \times M \times \overline{a + b x^m + x^{2m}}^{1-r} + \frac{B}{1+A} \times$$

$$\int \frac{x^{n-m} \dot{x}}{a + b x^m + x^{2m}}^r - \frac{C}{1+A} \times \int \frac{x^{n-2m} \dot{x}}{a + b x^m + x^{2m}}^r + \&c. = \frac{V}{1+A} \times$$

∫

$\int \frac{x^{n-1m} \dot{x}}{a+bx^m+x^{2m}} = \frac{W}{1+A} \times \frac{x^{n-1+1 \cdot m} \dot{x}}{a+bx^m+x^{2m}}^r$; hence if the two last fluents be given, we have the general law of continuation up to $\frac{x^n \dot{x}}{a+bx^m+x^{2m}}^r$ in the same manner as before.

III. In general, if we proceed as in the two last articles, we shall find $\int \frac{x^n \dot{x}}{a+bx^m+\&c. x^{tm}} = \frac{M}{P} \times \frac{x^{n-1+1 \cdot m} \dot{x}}{a+bx^m+\&c. x^{tm}}^{r-1} + \frac{A}{P} \times \int \frac{x^{n-m} \dot{x}}{a+bx^m+\&c. x^{tm}}^r + \frac{B}{P} \times \int \frac{x^{n-2m} \dot{x}}{a+bx^m+\&c. x^{tm}}^r + \&c. = \frac{T}{P} \times \int \frac{x^{n-1m} \dot{x}}{a+bx^m+\&c. x^{tm}}^r = \frac{V}{P} \times \int \frac{x^{n-1+1 \cdot m} \dot{x}}{a+bx^m+\&c. x^{tm}}^r + \&c.$ where the number of these last terms is t , and $M = \frac{x^{n-1m+1} \dot{x}}{n-1m+1} - \frac{Qx^{n-1+1 \cdot m-1} \dot{x}}{n-1+1 \cdot m+1} + \&c.$ omitting, as before, the terms at the end arising from the remainders. Hence if the last t fluents be given, we can by continuation find the required fluent.

Because the division of $\frac{x^n \dot{x}}{a+bx^m+\&c. x^{tm}}$ may be expressed by an ascending series $x^n \dot{x} - Qx^{n+m} \dot{x} + Rx^{n+2m} \dot{x} - \&c.$ it is manifest, that by the same method we may continue the fluents downwards as well as upwards.

IV. Let $\dot{F} = \frac{x^n \dot{x}}{1-x} = -x^{n-1} \dot{x} - x^{n-2} \dot{x} - x^{n-3} \dot{x} - \&c. - x^{n-r} \dot{x} + \frac{x^{n-r} \dot{x}}{1-x}$, then $F = -\frac{x^n}{n} - \frac{x^{n-1}}{n-1} - \frac{x^{n-2}}{n-2} - \&c. - \frac{x^{n-r+1}}{n-r+1} + W$, where W is the fluent of the last term. Now $\frac{x^n \dot{x}}{\sqrt{1-x^2}} = \frac{x^n \dot{x}}{1-x} \times \sqrt{\frac{1-x}{1+x}}$,

hence $\int \frac{x^n \dot{x}}{\sqrt{1-x^2}} = \int \frac{x^n \dot{x}}{1-x} \times \sqrt{\frac{1-x}{1+x}} = F \times \sqrt{\frac{1-x}{1+x}} + \int F \times$

$$\frac{\dot{x}}{\sqrt{1+x^2} \times 1+x} = F \times \sqrt{\frac{1-x}{1+x}} - \int \frac{x^n \dot{x}}{n \sqrt{1-x^2} \times 1+x} - \int \frac{x^{n-1} \dot{x}}{(n-1) \cdot \sqrt{1-x^2} \times 1+x}$$

$$- \int \frac{x^{n-2} \dot{x}}{(n-2) \cdot \sqrt{1-x^2} \times 1+x} - \&c. - \int \frac{x^{n-r+1} \dot{x}}{(n-r+1) \cdot \sqrt{1-x^2} \times 1+x} +$$

$$\int W \times \frac{\dot{x}}{\sqrt{1-x^2} \times 1+x}. \text{ But}$$

$$\frac{x^n \dot{x}}{1-x^2 \times 1+x} = \frac{x^{n-1} \dot{x}}{n \sqrt{1-x^2}} - \frac{x^{n-2} \dot{x}}{n \sqrt{1-x^2}} + \frac{x^{n-3} \dot{x}}{n \sqrt{1-x^2}} - \&c. + \frac{x^{n-r} \dot{x}}{n \sqrt{1-x^2}} + \frac{x^{n-r} \dot{x}}{n \sqrt{1-x^2} \times 1+x}$$

$$\frac{x^{n-1} \dot{x}}{\sqrt{1-x^2} \times 1+x} = \frac{x^{n-2} \dot{x}}{(n-1) \cdot \sqrt{1-x^2}} - \frac{x^{n-3} \dot{x}}{(n-1) \cdot \sqrt{1-x^2}} + \&c. + \frac{x^{n-r} \dot{x}}{(n-1) \cdot \sqrt{1-x^2}} + \frac{x^{n-r} \dot{x}}{(n-1) \cdot \sqrt{1-x^2} \times 1+x}$$

$$\frac{x^{n-2} \dot{x}}{\sqrt{1-x^2} \times 1+x} = \frac{x^{n-3} \dot{x}}{(n-2) \cdot \sqrt{1-x^2}} - \&c. + \frac{x^{n-r} \dot{x}}{(n-2) \cdot \sqrt{1-x^2}} + \frac{x^{n-r} \dot{x}}{(n-2) \cdot \sqrt{1-x^2} \times 1+x}$$

$$\&c. \qquad \&c. \qquad \&c.$$

$$\frac{x^{n-r+1} \dot{x}}{1 \cdot \sqrt{1-x^2} \times 1+x} = \frac{x^{n-r} \dot{x}}{(n-r+1) \cdot \sqrt{1+x^2}} - \frac{x^{n-r} \dot{x}}{(n-r+1) \cdot \sqrt{1-x^2} \times 1+x}.$$

Hence $\int \frac{x^n \dot{x}}{\sqrt{1-x^2}} = F \times \sqrt{\frac{1-x}{1+x}} - \frac{1}{n} \times \int \frac{x^{n-1} \dot{x}}{\sqrt{1-x^2}} + \frac{1}{n} - \frac{1}{n-1} \times$

$$\int \frac{x^{n-2} \dot{x}}{\sqrt{1-x^2}} - \frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-2} \times \int \frac{x^{n-3} \dot{x}}{\sqrt{1-x^2}} + \&c. \pm \frac{1}{n} \mp \frac{1}{n-1} \pm \&c.$$

$$- \frac{1}{n-r+1} \times \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2}} \mp \frac{1}{n} \pm \frac{1}{n-1} \mp \&c. + \frac{1}{n-r+1} \times \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2} \times 1+x}$$

$$+ \int W \times \frac{\dot{x}}{\sqrt{1-x^2} \times 1+x}. \text{ Now } \int W \times \frac{\dot{x}}{\sqrt{1-x^2} \times 1+x} = -W \times$$

$$\left[\frac{1-x}{1+x} \right]^{\frac{1}{2}} + \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2}}; \text{ also } \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2} \times 1+x} = -x^{n-r} \times \sqrt{\frac{1-x}{1+x}} +$$

$$\frac{1}{n-r} \times \int \frac{x^{n-r-1} \dot{x}}{\sqrt{1-x^2}} - \frac{1}{n-r} \times \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2}}; \text{ hence, by substituting}$$

these quantities in the two last terms, it is manifest that

$$-W \times \left[\frac{1-x}{1+x} \right]^{\frac{1}{2}} \text{ will be destroyed by the last term of}$$

F x

$F \times \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}}$ when we substitute for F its value; therefore, if we

put $M = -\frac{x^n}{n} - \frac{x^{n-1}}{n-1} - \frac{x^{n-2}}{n-2} - \&c. - \frac{x^{n-r+1}}{n-r+1}$, we shall have

$$\int \frac{x^n \dot{x}}{\sqrt{1+x}} = M \pm \frac{1}{n} \mp \frac{1}{n-1} \pm \&c. - \frac{1}{n-r+1} \times \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}} - \frac{1}{n} \times \int \frac{x^{n-1} \dot{x}}{\sqrt{1-x^2}}$$

$$+ \frac{1}{n} - \frac{1}{n-1} \times \int \frac{x^{n-2} \dot{x}}{\sqrt{1-x^2}} - \frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-2} \times \int \frac{x^{n-3} \dot{x}}{\sqrt{1-x^2}} + \&c.$$

$$\pm \frac{1}{n} \mp \frac{1}{n-1} \pm \&c. - \frac{1}{n-r+1} \times n-r+1 + 1 \times \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2}} \mp$$

$$\frac{1}{n} \pm \frac{1}{n-1} \mp \&c. + \frac{1}{n-r+1} \times n-r \times \int \frac{x^{n-r-1} \dot{x}}{\sqrt{1-x^2}}. \text{ Hence, if the}$$

two last fluents be given, we have the general law of conti-

nuation up to the fluent of $\frac{x^n \dot{x}}{\sqrt{1-x^2}}$, where the index of x with-

out the vinculum increases by unity each time. And in the

same manner we may (by increasing the index of x without by

m) find the fluent of $\frac{x^n \dot{x}}{\sqrt{a^{2m} - x^{2m}}}$ if we have given the fluents of

$$\frac{x^{n-r+m} \dot{x}}{\sqrt{a^{2m} - x^{2m}}} \text{ and } \frac{x^{n-r-1+m} \dot{x}}{\sqrt{a^{2m} - x^{2m}}}. \text{ Thus we have a general law of con-}$$

tinuation, where the index of x without is increased by half the index under the vinculum.

$$\text{V. Assume } \dot{F} = \frac{x^n \dot{x}}{x^m - b} = x^{n-m} \dot{x} + b x^{n-2m} \dot{x} + b^2 x^{n-3m} \dot{x} + \&c. +$$

$$\frac{b^r x^{n-rm} \dot{x}}{x^m - b}, \text{ then } F = \frac{x^{n-m+1}}{n-m+1} + \frac{b x^{n-2m+1}}{n-2m+1} + \frac{b^2 x^{n-3m+1}}{n-3m+1} + \&c. + W,$$

where W is put for the fluent of the last term. Now

$$\int x^n \dot{x} \sqrt{\frac{x^m - a}{x^m - b}} = \int \frac{x^n}{x^m - b} \times \sqrt{x^m - a} \times \sqrt{x^m - b} = F \times \sqrt{x^m - a} \times x^m + b$$

$$\begin{aligned}
 & - \int F \times \frac{2mx^{2m-1}\dot{x} - \overline{a+b} \cdot mx^{m-1}\dot{x}}{2\sqrt{x^m - a} \times x^m - b} = (\text{by substituting for } F \text{ its} \\
 & \text{value in the latter quantity, and putting } A, B, C, \&c. \text{ for} \\
 & \text{the co-efficients which arise therefrom}) F \times \sqrt{x^m - a} \times x^m - b - \\
 & A \int \frac{x^n \dot{x}}{\sqrt{x^m - a} \times x^m - b} - B \int \frac{x^{n-m} \dot{x}}{\sqrt{x^m - a} \times x^m - b} - C \int \frac{x^{n-2m} \dot{x}}{\sqrt{x^m - a} \times x^m - b} - \\
 & \&c. - \int W \times \frac{2mx^{2m-1}\dot{x} - \overline{a+b} \cdot mx^{m-1}\dot{x}}{2\sqrt{x^m - a} \times x^m - b}. \text{ Now } \frac{x^n \dot{x}}{\sqrt{x^m - a} \times x^m - b} = \\
 & \frac{x^n \dot{x}}{x^m - a} \sqrt{\frac{x^m - a}{x^m - b}}, \frac{x^{n-m} \dot{x}}{\sqrt{x^m - a} \times x^m - b} = \frac{x^{n-m} \dot{x}}{x^m - a} \sqrt{\frac{x^m - a}{x^m - b}}, \frac{x^{n-2m} \dot{x}}{\sqrt{x^m - a} \times x^m - b} = \frac{x^{n-2m} \dot{x}}{x^m - a} \\
 & \sqrt{\frac{x^m - a}{x^m - b}}, \&c. \text{ But}
 \end{aligned}$$

$$\begin{aligned}
 \frac{x^n \dot{x}}{x^m - a} &= x^{n-m} \dot{x} + ax^{n-2m} \dot{x} + a^2 x^{n-3m} \dot{x} + \&c. + a^{r-1} x^{n-rm} \dot{x} + \frac{a^r x^{n-rm} \dot{x}}{x^m - a} \\
 \frac{x^{n-m} \dot{x}}{x^m - a} &= x^{n-2m} \dot{x} + ax^{n-3m} \dot{x} + \&c. + a^{r-2} x^{n-rm} \dot{x} + \frac{a^{r-1} x^{n-rm} \dot{x}}{x^m - a} \\
 \frac{x^{n-2m} \dot{x}}{x^m - a} &= x^{n-3m} \dot{x} + \&c. + a^{r-3} x^{n-rm} \dot{x} + \frac{a^{r-2} x^{n-rm} \dot{x}}{x^m - a} \\
 \&c. & \qquad \qquad \&c. \qquad \qquad \&c.
 \end{aligned}$$

$$\begin{aligned}
 \text{Hence, by substitution, } \int \frac{x^n \dot{x}}{\sqrt{x^m - a} \times x^m - b} &= F \times \sqrt{x^m - a} \times x^m - b \\
 &- A \int \frac{x^{n-m} \dot{x}}{\sqrt{x^m - a} \times x^m - b} - \overline{Aa + B} \times \int \frac{x^{n-2m} \dot{x}}{\sqrt{x^m - a} \times x^m - b} - \overline{Aa^2 + Ba + C} \\
 &\times \int \frac{x^{n-3m} \dot{x}}{\sqrt{x^m - a} \times x^m - b} - \&c. - \overline{Aa^{r-1} + Ba^{r-2} + Ca^{r-3} + \&c.} \times \int \\
 &\times \int \frac{x^{n-rm} \dot{x}}{\sqrt{x^m - a} \times x^m - b} - \overline{Aa^r + Ba^{r-1} + Ca^{r-2} + \&c.} \times \int \frac{x^{n-rm} \dot{x}}{\sqrt{x^m - a} \times x^m - b} \\
 &- \int W \times \frac{2mx^{2m-1}\dot{x} - \overline{a+b} \cdot mx^{m-1}\dot{x}}{2\sqrt{x^m - a} \times x^m - b}. \text{ But}
 \end{aligned}$$

$$- \int W \times \frac{2mx^{2m-1}\dot{x} - \overline{+b} \cdot mx^{m-1}\dot{x}}{2\sqrt{x^m-a} \times x^m-b} \text{ is } - W \times \sqrt{x^m-a} \times x^m-b$$

$$+ b^r \times \int x^{n-rm}\dot{x} \sqrt{\frac{x^m-a}{x^m-b}}; \text{ hence, by substituting this for the last}$$

term, it is manifest, that $-W \times \sqrt{x^m-a} \times x^m-b$ will be destroyed by the last term of $F \times \sqrt{x^m-a} \times x^m-b$ when we substitute for F its value; therefore, if we put $M = \frac{x^{n-m+1}}{n-m+1} +$

$$\frac{bx^{n-2m+1}}{n-2m+1} + \&c. + \frac{b^{r-1}x^{n-rm+1}}{n-rm+1}, \text{ we have } \int x^n\dot{x} \sqrt{\frac{x^m-a}{x^m-b}} = M \times$$

$$\sqrt{x^m-a} \times x^m-b - A \int x^{n-m}\dot{x} \sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa} + B \times \int x^{n-2m}\dot{x}$$

$$\sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa^2 + Ba} + C \times \int x^{n-3m}\dot{x} \sqrt{\frac{x^m-a}{x^m-b}} - \&c. -$$

$$\overline{Aa^{r-1} + Ba^{r-2} + \&c.} - b^r \times \int x^{n-rm}\dot{x} \sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa^r + Ba^{r-1} + Ca^{r-2}}$$

$$+ \&c. \times \int \frac{x^{n-rm}\dot{x}}{\sqrt{x^m-a} \times x^m-b}. \text{ Hence if the last two fluents be}$$

given, we have the general law of continuation up to the

$$\text{fluent of } x^n\dot{x} \sqrt{\frac{x^m-a}{x^m-b}}.$$

The utility of finding fluents by continuation was manifest to Sir ISAAC NEWTON, who first proposed it; and since his time some of the most eminent mathematicians have employed much of their attention upon it. The method which I have investigated and exemplified in this Paper I offer as being entirely new; and at the same time it not only exhibits, at once, the general law up to the required fluent, but also appears, from some of the instances here given, to be more extensive and convenient in its application than any method hitherto offered.

The

The general resolution of the given fluxion into a series of fluxions of the same kind, where the index of the unknown quantity without the vinculum keeps decreasing or increasing either by the index under or by half the index, has not, that I know of, before been given; which furnishes us at once not only with a very easy method of *continuing* fluents, but also points out a very simple method of investigating the fluent of the given fluxion *without* continuation. For if $\int \dot{A} = p + b \int \dot{B} + c \int \dot{C} + d \int \dot{D} + \&c. \int \dot{B} = p' + c' \int \dot{C} + d' \int \dot{D} + \&c. \int \dot{C} = p'' + d'' \int \dot{D} + \&c. \&c. \&c.$ then if for $\int \dot{B}$, $\int \dot{C}$, &c. &c. we substitute their respective values, we shall get a general series for $\int \dot{A}$ without continuation. The extent of any new method is, at first, seldom obvious; and how far that which is here proposed may be successfully employed in other cases will best appear from its application. Different methods will always be found to have their uses in particular cases; for where one becomes impracticable another will often be found to succeed; and I hope that which is here offered will contribute something towards facilitating the investigation of fluents.

